

Lagrangian and Hamiltonian dynamics of submanifolds

G.Giachetta, L.Mangiarotti¹, G.Sardanashvily²

¹ Department of Mathematics and Informatics, University of Camerino, 62032 Camerino (MC), Italy

² Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Abstract Submanifolds of a manifold are described as sections of a certain fiber bundle that enables one to consider their Lagrangian and (polysymplectic) Hamiltonian dynamics as that of a particular classical field theory. In particular, their Lagrangians and Hamiltonians must satisfy rather restrictive Noether identities. For instance, this is the case of relativistic mechanics and classical string theory.

1 Introduction

As is well known, fiber bundles and jet manifolds of their sections provide an adequate mathematical formulation of classical field theory. In particular, field Lagrangians and their Euler–Lagrange operators are algebraically described as elements of the variational bicomplex [1, 2, 3]. This description is extended to Lagrangian theory of odd fields [4, 5, 6]. The Hamiltonian counterpart of first-order Lagrangian theory on fiber bundles is covariant Hamiltonian formalism developed in multisymplectic, polysymplectic, and Hamilton – De Donder) variants (see, e.g., [7, 8, 9, 10, 11]).

Jets of sections of fiber bundles are particular jets of submanifolds. Namely, a space of jets of submanifolds admits a cover by charts of jets of sections [7, 12, 13]. Three-velocities in relativistic mechanics exemplify first order jets of submanifolds [14, 15]. A problem is that differential forms on jets of submanifolds do not constitute a variational bicomplex because horizontal forms (e.g., Lagrangians) are not preserved under coordinate transformations.

We consider n -dimensional submanifolds of an m -dimensional smooth real manifold Z , and associate to them sections of a trivial fiber bundle $Z_Q = Q \times Z \rightarrow Q$, where Q is some n -dimensional manifold. Here, we restrict our consideration to first order jets of submanifolds, and state their relation to jets of sections of the fiber bundle $Z_Q \rightarrow Q$ (the formulas (17), (19), and Proposition 1). This relation fails to be one-to-one correspondence. The ambiguity contains, e.g., diffeomorphisms of Q . Then Lagrangian and (polysymplectic) Hamiltonian formalism on a fiber bundle $Z_Q \rightarrow Q$ is developed in a standard way, but Lagrangians

and Hamiltonians are required to be variationally invariant under the above mentioned diffeomorphisms of Q . This invariance however leads to rather restrictive Noether identities (31) and (48) which these Lagrangians and Hamiltonians must satisfy, unless other fields are introduced.

In a different way, one can choose some subbundle of the fiber bundle $Z_Q \rightarrow Q$ in order to avoid the above mentioned ambiguity between jets of subbundles of Z and jets of sections of $Z_Q \rightarrow Q$. Since such a subbundle itself need not be a jet manifold of some fiber bundle, it is a nonholonomic constraint. For instance, this is the case of relativistic mechanics, phrased in terms of four-velocities.

2 Jets of submanifolds

Given an m -dimensional smooth real manifold Z , a k -order jet of n -dimensional submanifolds of Z at a point $z \in Z$ is defined as the equivalence class $j_z^k S$ of n -dimensional imbedded submanifolds of Z through z which are tangent to each other at z with order $k \geq 0$. Namely, two submanifolds $i_S : S \hookrightarrow Z$, $i_{S'} : S' \hookrightarrow Z$ through a point $z \in Z$ belong to the same equivalence class $j_z^k S$ iff the images of the k -tangent morphisms

$$T^k i_S : T^k S \hookrightarrow T^k Z, \quad T^k i_{S'} : T^k S' \hookrightarrow T^k Z$$

coincide with each other. The set $J_n^k Z = \bigcup_{z \in Z} j_z^k S$ of k -order jets is a finite-dimensional real smooth manifold. One puts $J_n^0 Z = Z$. If $k > 0$, let $Y \rightarrow X$ be an m -dimensional fiber bundle over an n -dimensional base X and $J^k Y$ the k -order jet manifold of sections of $Y \rightarrow X$ (or, shortly, the jet manifold of $Y \rightarrow X$). Given an imbedding $\Phi : Y \rightarrow Z$, there is the natural injection

$$J^k \Phi : J^k Y \rightarrow J_n^k Z, \quad j_x^k s \mapsto [\Phi \circ s]_{\Phi(s(x))}^k,$$

where s are sections of $Y \rightarrow X$. This injection defines a chart on $J_n^k Z$. These charts provide a manifold atlas of $J_n^k Z$.

Here, we restrict our consideration to first order jets of submanifolds. There is obvious one-to-one correspondence

$$\zeta : j_z^1 S \mapsto V_{j^1 S} \subset T_z Z \tag{1}$$

between the jets $j_z^1 S$ at a point $z \in Z$ and the n -dimensional vector subspaces of the tangent space $T_z Z$ of Z at z . It follows that $J_n^1 Z$ is a fiber bundle

$$\rho : J_n^1 Z \rightarrow Z \tag{2}$$

in Grassmann manifolds. It possesses the following coordinate atlas.

Let $\{(U; z^\mu)\}$ be a coordinate atlas of Z . Though $J_n^0 Z = Z$, let us provide $J_m^0 Z$ with the atlas obtained by replacing every chart (U, z^A) of Z with the

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

charts on U which correspond to different partitions of (z^A) in collections of n and $m-n$ coordinates

$$(U; x^a, y^i), \quad a = 1, \dots, n, \quad i = 1, \dots, m-n. \quad (3)$$

The transition functions between the coordinate charts (3) of $J_n^0 Z$ associated with a coordinate chart (U, z^A) of Z reduce to an exchange between coordinates x^a and y^i . Transition functions between arbitrary coordinate charts of the manifold $J_n^0 Z$ take the form

$$x'^a = x'^a(x^b, y^k), \quad y'^i = y'^i(x^b, y^k). \quad (4)$$

Given an atlas of coordinate charts (3) – (4) of the manifold $J_n^0 Z$, the first order jet manifold $J_n^1 Z$ is endowed with the coordinate charts

$$(\rho^{-1}(U) = U \times \mathbb{R}^{(m-n)n}; x^a, y^i, y_a^i), \quad (5)$$

possessing the following transition functions. With respect to the coordinates (5) on the jet manifold $J_n^1 Z$ and the induced fiber coordinates (\dot{x}^a, \dot{y}^i) on the tangent bundle TZ , the above mentioned correspondence ζ (1) reads

$$\zeta : (y_a^i) \mapsto \dot{x}^a(\partial_a + y_a^i(j_z^1 S)\partial_i).$$

It implies the relations

$$y_a'^j = \left(\frac{\partial y'^j}{\partial y^k} y_b^k + \frac{\partial y'^j}{\partial x^b}\right) \left(\frac{\partial x^b}{\partial y'^i} y_a^i + \frac{\partial x^b}{\partial x'^a}\right), \quad (6)$$

$$\left(\frac{\partial x^b}{\partial y'^i} y_a^i + \frac{\partial x^b}{\partial x'^a}\right) \left(\frac{\partial x'^c}{\partial y^k} y_b^k + \frac{\partial x'^c}{\partial x^b}\right) = \delta_a^c, \quad (7)$$

which jet coordinates y_a^i must satisfy under coordinate transformations (4). Let consider a nondegenerate $n \times n$ matrix M with the entries

$$M_b^c = \left(\frac{\partial x'^c}{\partial y^k} y_b^k + \frac{\partial x'^c}{\partial x^b}\right).$$

Then the relations (7) lead to the equalities

$$\left(\frac{\partial x^b}{\partial y'^i} y'_a{}^i + \frac{\partial x^b}{\partial x'^a}\right) = (M^{-1})^b_a.$$

Hence, we obtain the transformation law of first order jet coordinates

$$y'^j_a = \left(\frac{\partial y'^j}{\partial y^k} y^k_b + \frac{\partial y'^j}{\partial x^b}\right) (M^{-1})^b_a. \quad (8)$$

For instance, these are the Lorentz transformation of three-velocities in relativistic mechanics. In particular, if coordinate transition functions x'^a (4) are independent of coordinates y^k , the transformation law (8) comes to the familiar transformations of jets of sections.

A glance at the transformations (8) shows that, in contrast with a fiber bundle of jets of sections, the fiber bundle (2) is not affine. In particular, one generalizes the notion of a connection on fiber bundles and treat global sections of the jet bundle (2) as preconnections [13]. However, a global section of this bundle need not exist ([16], Theorem 27.18).

Given a coordinate chart (5) of $J^1_n Z$, one can regard $\rho^{-1}(U) \subset J^1_n Z$ as the first order jet manifold $J^1 U$ of sections of the fiber bundle

$$U \ni (x^a, y^i) \rightarrow (x^a) \in U_X. \quad (9)$$

The graded differential algebra $\mathcal{O}^*(\rho^{-1}(U))$ of exterior forms on $\rho^{-1}(U)$ is generated by horizontal forms dx^a and contact forms $dy^i - y^i_a dx^a$. Coordinate transformations (4) and (8) preserve the ideal of contact forms, but horizontal forms are not transformed into horizontal forms, unless coordinate transition functions x'^a (4) are independent of coordinates y^k . Therefore, one can develop first order Lagrangian formalism with a Lagrangian $L = \mathcal{L} d^n x$ on a coordinate chart $\rho^{-1}(U)$, but this Lagrangian fails to be globally defined on $J^1_n Z$.

In order to overcome this difficulty, let us consider an above mentioned product $Z_Q = Q \times Z$ of Z and an n -dimensional real smooth manifold Q . We have a trivial fiber bundle

$$\pi : Z_Q = Q \times Z \rightarrow Q, \quad (10)$$

whose trivialization throughout holds fixed. This fiber bundle is provided with an atlas of coordinate charts

$$(U_Q \times U; q^\mu, x^a, y^i), \quad (11)$$

where $(U; x^a, y^i)$ are the above mentioned coordinate charts (3) of the manifold $J^0_n Z$. The coordinate charts (11) possess transition functions

$$q'^\mu = q^\mu(q^\nu), \quad x'^a = x'^a(x^b, y^k), \quad y'^i = y'^i(x^b, y^k). \quad (12)$$

Let $J^1 Z_Q$ be the first order jet manifold of the fiber bundle (10). Since the trivialization (10) is fixed, it is a vector bundle $\pi^1 : J^1 Z_Q \rightarrow Z_Q$ isomorphic to the tensor product

$$J^1 Z_Q = T^*Q \otimes_{Q \times Z} TZ \quad (13)$$

of the cotangent bundle T^*Q of Q and the tangent bundle TZ of Z over Z_Q .

Given a coordinate atlas (11) - (12) of Z_Q , the jet manifold $J^1 Z_Q$ is endowed with the coordinate charts

$$((\pi^1)^{-1}(U_Q \times U) = U_Q \times U \times \mathbb{R}^{mn}; q^\mu, x^a, y^i, x_\mu^a, y_\mu^i), \quad (14)$$

possessing transition functions

$$x'_\mu{}^a = \left(\frac{\partial x'^a}{\partial y^k} y_\nu^k + \frac{\partial x'^a}{\partial x^b} x_\nu^b \right) \frac{\partial q^\nu}{\partial q'^\mu}, \quad y'_\mu{}^i = \left(\frac{\partial y'^i}{\partial y^k} y_\nu^k + \frac{\partial y'^i}{\partial x^b} x_\nu^b \right) \frac{\partial q^\nu}{\partial q'^\mu}. \quad (15)$$

Relative to coordinates (14), the isomorphism (13) takes the form

$$(x_\mu^a, y_\mu^i) \rightarrow dq^\mu \otimes (x_\mu^a \partial_a + y_\mu^i \partial_i).$$

Obviously, a jet $(q^\mu, x^a, y^i, x_\mu^a, y_\mu^i)$ of sections of the fiber bundle (10) defines some jet of n -dimensional subbundles of the manifold $\{q\} \times Z$ through a point $(x^a, y^i) \in Z$ if an $m \times n$ matrix with the entries x_μ^a, y_μ^i is of maximal rank n . This property is preserved under the coordinate transformations (15). An element of $J^1 Z_Q$ is called regular if it possesses this property. Regular elements constitute an open subbundle of the jet bundle $J^1 Z_Q \rightarrow Z_Q$.

Since regular elements of $J^1 Z_Q$ characterize jets of submanifolds of Z , one hopes to describe the dynamics of submanifolds of a manifold Z as that of sections of the fiber bundle (10). For this purpose, let us refine the relation between elements of the jet manifolds $J_n^1 Z$ and $J^1 Z_Q$.

Let us consider the manifold product $Q \times J_n^1 Z$. Of course, it is a bundle over Z_Q . Given a coordinate atlas (11) - (12) of Z_Q , this product is endowed with the coordinate charts

$$(U_Q \times \rho^{-1}(U) = U_Q \times U \times \mathbb{R}^{(m-n)n}; q^\mu, x^a, y^i, y_a^i), \quad (16)$$

possessing transition functions (8). Let us assign to an element (q^μ, x^a, y^i, y_a^i) of the chart (16) the elements $(q^\mu, x^a, y^i, x_\mu^a, y_\mu^i)$ of the chart (14) whose coordinates obey the relations

$$\boxed{y_a^i x_\mu^a = y_\mu^i}. \quad (17)$$

These elements make up an n^2 -dimensional vector space. The relations (17) are maintained under coordinate transformations (12) and the induced transformations of the charts (14) and (16) as follows:

$$\begin{aligned} y_a'^i x_\mu'^a &= \left(\frac{\partial y'^i}{\partial y^k} y_c^k + \frac{\partial y'^i}{\partial x^c} \right) (M^{-1})_a^c \left(\frac{\partial x'^a}{\partial y^k} y_\nu^k + \frac{\partial x'^a}{\partial x^b} x_\nu^b \right) \frac{\partial q^\nu}{\partial q'^\mu} = \\ &= \left(\frac{\partial y'^i}{\partial y^k} y_c^k + \frac{\partial y'^i}{\partial x^c} \right) (M^{-1})_a^c \left(\frac{\partial x'^a}{\partial y^k} y_b^k + \frac{\partial x'^a}{\partial x^b} \right) x_\nu^b \frac{\partial q^\nu}{\partial q'^\mu} = \\ &= \left(\frac{\partial y'^i}{\partial y^k} y_b^k + \frac{\partial y'^i}{\partial x^b} \right) x_\nu^b \frac{\partial q^\nu}{\partial q'^\mu} = \left(\frac{\partial y'^i}{\partial y^k} y_\nu^k + \frac{\partial y'^i}{\partial x^b} x_\nu^b \right) \frac{\partial q^\nu}{\partial q'^\mu} = y_\mu'^i. \end{aligned}$$

Thus, one can associate

$$\zeta' : (q^\mu, x^a, y^i, y_a^i) \mapsto \{(q^\mu, x^a, y^i, x_\mu^a, y_\mu^i) \mid y_a^i x_\mu^a = y_\mu^i\} \quad (18)$$

to each element of the manifold $Q \times J_n^1 Z$ an n^2 -dimensional vector space in the jet manifold $J^1 Z_Q$. This is a subspace of elements $x_\mu^a dq^\mu \otimes (\partial_a + y_a^i \partial_i)$ of a fiber of the tensor bundle (13) at a point (q^μ, x^a, y^i) . This subspace always contains regular elements, e.g., whose coordinates x_μ^a form a nondegenerate $n \times n$ matrix.

Conversely, given a regular element $j_z^1 s$ of $J^1 Z_Q$, there is a coordinate chart (14) such that coordinates x_μ^a of $j_z^1 s$ constitute a nondegenerate matrix, and $j_z^1 s$ defines a unique element of $Q \times J_n^1 Z$ by the relations

$$\boxed{y_a^i = y_\mu^i (x^{-1})_a^\mu} \quad (19)$$

For instance, this is the well-known relation between three- and four-velocities in relativistic mechanics.

Thus, we have shown the following. Let (q^μ, z^A) further be arbitrary coordinates on the product Z_Q (10) and (q^μ, z^A, z_μ^A) the corresponding coordinates on the jet manifold $J^1 Z_Q$. In these coordinates, an element of $J^1 Z_Q$ is regular if an $m \times n$ matrix with the entries z_μ^A is of maximal rank n .

Proposition 1. (i) Any jet of submanifolds through a point $z \in Z$ defines some (but not unique) jet of sections of the fiber bundle Z_Q (10) through a point $q \times z$ for any $q \in Q$ in accordance with the relations (17).

(ii) Any regular element of $J^1 Z_Q$ defines a unique element of the jet manifold $J_n^1 Z$ by means of the relations (19). However, nonregular elements of $J^1 Z_Q$ can correspond to different jets of submanifolds.

(iii) Two elements (q^μ, z^A, z_μ^A) and $(q^\mu, z^A, z_\mu'^A)$ of $J^1 Z_Q$ correspond to the same jet of submanifolds if $z_\mu'^A = M_\mu^\nu z_\nu^A$, where M is some matrix, e.g., it comes from a diffeomorphism of Q .

Basing on this result, we can describe the dynamics of n -dimensional submanifolds of a manifold Z as that of sections of the fiber bundle $Q \times Z \rightarrow Q$ for some n -dimensional manifold Q .

3 Lagrangian dynamics of submanifolds

Let Z_Q be a fiber bundle (10) coordinated by (q^μ, z^A) with transition functions $q'^\mu(q^\nu)$ and $z'^A(z^B)$. Then the first order jet manifold $J^1 Z_Q$ of this fiber bundle is provided with coordinates (q^μ, z^A, z_μ^A) possessing transition functions

$$z_\mu'^A = \frac{\partial z'^A}{\partial z^B} \frac{\partial q^\nu}{\partial q'^\mu} z_\nu^B.$$

Let us recall the notation of contact forms $\theta^A = dz^A - z_\mu^A dq^\mu$, operators of total derivatives

$$d_\mu = \partial_\mu + z_\mu^A \partial_A + z_{\mu\nu}^A \partial_A^\nu,$$

the total differential $d_H(\phi) = dq^\mu \wedge d_\mu(\phi)$ acting on exterior forms ϕ on $J^1 Z_Q$, and the horizontal projection $h_0(\theta^A) = 0$.

A first order Lagrangian in Lagrangian formalism on a fiber bundle $Z_Q \rightarrow Q$ is defined as a horizontal density

$$L = \mathcal{L}(z^A, z_\mu^A) \omega, \quad \omega = dq^1 \wedge \cdots \wedge dq^n, \quad (20)$$

on the jet manifold $J^1 Z_Q$. The corresponding Euler–Lagrange operator reads

$$\delta L = \mathcal{E}_A dz^A \wedge \omega, \quad \mathcal{E}_A = \partial_A \mathcal{L} - d_\mu \partial_A^\mu \mathcal{L}. \quad (21)$$

It yields the Euler–Lagrange equations

$$\mathcal{E}_A = \partial_A \mathcal{L} - d_\mu \partial_A^\mu \mathcal{L} = 0. \quad (22)$$

Let $u = u^\mu \partial_\mu + u^A \partial_A$ be a vector field on Z_Q . Its jet prolongation onto $J^1 Z_Q$ reads

$$u = u^\mu \partial_\mu + u^A \partial_A + (d_\mu u^A - z_\nu^A d_\mu u^\nu) \partial_A^\mu. \quad (23)$$

It admits the vertical splitting

$$u = u_H + u_V = u^\mu d_\mu + [(u^A - u^\nu z_\nu^A) \partial_A + d_\mu(u^A - z_\nu^A u^\nu) \partial_A^\mu]. \quad (24)$$

The Lie derivative $\mathbf{L}_u L$ of a Lagrangian L along a vector field u obeys the first variational formula

$$\mathbf{L}_u L = u_V \rfloor \delta L + d_H(h_0(u \rfloor H_L)) = ((u^A - u^\mu z_\mu^A) \mathcal{E}_A + d_\mu \mathfrak{J}^\mu) \omega, \quad (25)$$

where

$$H_L = L + \partial_A^\mu \mathcal{L} \theta^A \wedge \omega_\mu, \quad \omega_\mu = \partial_\mu \rfloor \omega, \quad (26)$$

is the Poincaré–Cartan form and

$$\mathfrak{J} = \mathfrak{J}^\mu \omega_\mu = (\partial_A^\mu \mathcal{L} (u^A - u^\nu z_\nu^A) + u^\mu \mathcal{L}) \omega_\mu \quad (27)$$

is the Noether current. A vector field u is called a variational symmetry of a Lagrangian L if the Lie derivative (25) is d_H -exact, i.e., $\mathbf{L}_u L = d_H \sigma$. In this case, there is the weak conservation law $0 \approx d_H(\mathfrak{J} - \sigma)$ on the shell (22).

One can show that a vector field u (23) is a variational symmetry only if it is projected onto Q , i.e., its components u^μ are functions on Q , and iff its vertical part u_V (24) is a variational symmetry. In a general setting, one deals with generalized vector fields u depending on parameter functions $\xi^r(q^\nu)$, their derivatives $\partial_{\lambda_1 \dots \lambda_k} \xi^r$, and higher order jets $z_{\lambda_1 \dots \lambda_k}^A$ [5, 17]. A vertical variational symmetry depending on parameters is called a gauge symmetry. Here we restrict our consideration to gauge symmetries u which are linear in parameters and their first derivatives, i.e.,

$$u = u^A \partial_A + d_\mu u_A \partial_A^\mu, \quad u^A = u_r^A \xi^r + u_r^{A,\mu} \partial_\mu \xi^r, \quad (28)$$

where u^A are functions of q^μ , z^B and the jets $z_{\mu_1 \dots \mu_k}^B$ of bounded jet order $k < N$. By virtue of the Noether second theorem [6, 17, 18], u (28) is a gauge symmetry of a Lagrangian L (20) iff the variational derivatives \mathcal{E}_A (21) of L obey the Noether identities

$$(u_r^A - d_\mu u_r^{A,\mu}) \mathcal{E}_A - u_r^{A,\mu} d_\mu \mathcal{E}_A = 0. \quad (29)$$

For instance, let us consider an arbitrary vector field $u = u^\mu(q^\nu) \partial_\mu$ on Q . It is an infinitesimal generator of a one-parameter group of local diffeomorphisms of Q . Since $Z_Q \rightarrow Q$ is a trivial bundle, this vector field gives rise to a vector field $u = u^\mu \partial_\mu$ on Z_Q , and its jet prolongation (23) onto $J^1 Z_Q$ reads

$$u = u^\mu \partial_\mu - z_\nu^A \partial_\mu u^\nu \partial_A^\mu = u^\mu d_\mu + [-u^\nu z_\nu^A \partial_A - d_\mu (u^\nu z_\nu^A) \partial_A^\mu]. \quad (30)$$

One can regard it as a generalized vector field depending on parameter functions $u^\mu(q^\nu)$. In accordance with Proposition 1, it seems reasonable to require that, in order to describe jets of submanifolds of Z , a Lagrangian L on $J^1 Z_Q$ is independent on coordinates of Q and must be variationally invariant under u (30) or, equivalently, its vertical part

$$u_V = -u^\nu z_\nu^A \partial_A - d_\mu(u^\nu z_\nu^A) \partial_A^\mu.$$

Then the variational derivatives of this Lagrangian obey irreducible Noether identities (29) which read

$$\boxed{z_\nu^A \mathcal{E}_A = 0.} \quad (31)$$

These Noether identities are rather restrictive, unless other fields are introduced.

In order to extend Lagrangian formalism on Z_Q to other fields, one can use the following two constructions: (i) a bundle product $Z_Q \times Z'$ of Z_Q and some bundle $Z' \rightarrow Q$ bundle over Q (e.g., a tensor bundle $\overset{k}{\otimes} TQ \overset{r}{\otimes} T^*Q$), (ii) a bundle $E \rightarrow Z$ and its pull-back E_Q onto $Q \times Z$ which is a composite bundle

$$E_Q = Q \times E \rightarrow Z_Q \rightarrow Q. \quad (32)$$

Let $E \rightarrow Z$ be provided with bundle coordinates (z^A, s^i) . Its pull-back E_Q onto Z_Q possesses coordinates (q^μ, z^A, s^i) . Accordingly, the pull-back $Q \times J^1 E$ of the first order jet manifold $J^1 E$ of $E \rightarrow Z$ onto Z_Q is endowed with coordinates (q^μ, z^A, s^i, s_A^i) . It is a subbundle $Q \times J^1 E \subset J^1 E_Q$ of the first order jet manifold $J^1 E_Q \rightarrow Z_Q$ of the fiber bundle $E_Q \rightarrow Z_Q$. This subbundle consists of jets of sections of $E_Q \rightarrow Z_Q$ which are the pull-back of sections of $E \rightarrow Z$. Given a composite fiber bundle (32), there is the canonical bundle morphism

$$\gamma : J^1 Z_Q \times_{Z_Q} J^1 E_Q \xrightarrow{Z_Q} J_Q^1 E_Q \quad (33)$$

of the bundle product of jet manifolds $J^1 Z_Q, J^1 E_Q$ of the bundles $Z_Q \rightarrow Q, E_Q \rightarrow Z_Q$ to the first order jet manifold $J_Q^1 E_Q$ of the fiber bundle $E_Q \rightarrow Q$ [7, 21, 22]. The jet manifold $J_Q^1 E_Q$ is coordinated by $(q^\mu, z^A, s^i, z_\mu^A, s_\mu^i)$. Restricted to $Q \times J^1 E \subset J^1 E_Q$, the morphism (33) takes the coordinate form

$$(z_\mu^A, s_\mu^i) = \gamma(z_\mu^A, s_A^i) = (z_\mu^A, s_A^i z_\mu^A). \quad (34)$$

Due to the morphism (34), any connection

$$\Gamma = dz^A \otimes (\partial_A + \Gamma_A^i(z^B, s^j) \partial_i) \quad (35)$$

on a fiber bundle $E \rightarrow Z$ yields the covariant derivative

$$D_\mu s^i = s^i_\mu - \Gamma_A^i(z^B, s^j) z^A_\mu \quad (36)$$

on the composite bundle $E_Q \rightarrow Q$.

Given a fiber bundle $Z_Q \times Z'$ or a composite fiber bundle E_Q (32), an extended Lagrangian is defined on the jet manifolds $J^1 Z_Q \times J^1 Z'$ or $J^1_Q E_Q$, respectively. For instance, any horizontal n -form

$$\frac{1}{n!} \phi_{A_1 \dots A_n} dz^{A_1} \wedge \dots \wedge dz^{A_n}$$

on $E \rightarrow Z$ yields a horizontal density

$$\frac{1}{n!} \phi_{A_1 \dots A_n} z^{A_1}_{\mu_1} \dots z^{A_n}_{\mu_n} dq^{\mu_1} \wedge \dots \wedge dq^{\mu_n}$$

on $J^1_Q E_Q \rightarrow Q$ which may contribute to a Lagrangian (see, e.g., a relativistic particle in the presence of an electromagnetic field).

4 Hamiltonian dynamics of submanifolds

Here, we follow polysymplectic Hamiltonian formalism which aims to describe field systems with nonregular Lagrangians [7, 8, 19, 20]. Lagrangian and polysymplectic Hamiltonian formalisms are equivalent in the case of hyperregular Lagrangians, but a nonregular Lagrangian admits different associated Hamiltonians, if any. At the same time, there is a comprehensive relation between these formalisms in the case of almost-regular Lagrangians.

Given a fiber bundle Z_Q (10) and its vertical cotangent bundle $V^* Z_Q$, let us consider the fiber bundle

$$\Pi = V^* Z_Q \wedge \binom{n-1}{n} T^* Q. \quad (37)$$

It plays the role of a momentum phase space of covariant Hamiltonian field theory. Given coordinates (q^μ, z^A) on Z_Q , this fiber bundle is coordinated by (q^μ, z^A, p_A^μ) , where p_A^μ are treated as coordinates of momenta. It is provided with the canonical polysymplectic form

$$\Omega_\Pi = dp_A^\mu \wedge dz^A \otimes \partial_\mu. \quad (38)$$

Every Lagrangian L on the jet manifold $J^1 Z_Q$ yields the Legendre map

$$\hat{L} : J^1 Z_Q \xrightarrow{Z_Q} \Pi, \quad p_A^\mu \circ \hat{L} = \partial_A^\mu \mathcal{L}, \quad (39)$$

whose range $N_L = \widehat{L}(J^1 Z_Q)$ is called the Lagrangian constraint space. A Lagrangian L is called hyperregular (resp. regular), if the Legendre map (39) is a diffeomorphism (resp. local diffeomorphism, i.e., of maximal rank). A Lagrangian L is said to be almost-regular if the Lagrangian constraint space is a closed imbedded subbundle $i_N : N_L \rightarrow \Pi$ of the Legendre bundle $\Pi \rightarrow Z_Q$ and the surjection $\widehat{L} : J^1 Z_Q \rightarrow N_L$ is a fibered manifold possessing connected fibers.

A multisymplectic momentum phase space is the homogeneous Legendre bundle

$$\overline{\Pi} = T^* Z_Q \wedge (\wedge^{n-1} T^* Q),$$

coordinated by (q^μ, z^A, p_A^μ, p) . It is endowed with the canonical multisymplectic form

$$\Xi = p\omega + p_A^\mu dz^A \wedge \omega_\mu.$$

There is a trivial one-dimensional bundle $\overline{\Pi} \rightarrow \Pi$. Then a Hamiltonian \mathcal{H} on the momentum phase space Π (37) is defined as a section $p = -\mathcal{H}$ of this fiber bundle. The pull-back of the multisymplectic form Ξ onto Π by a Hamiltonian \mathcal{H} is a Hamiltonian form

$$H = p_A^\mu dz^A \wedge \omega_\mu - \mathcal{H}\omega \quad (40)$$

on Π . The corresponding Hamilton equations with respect to the polysymplectic form Ω_Π (38) read

$$z_\mu^A - \partial_\mu^A \mathcal{H} = 0, \quad -p_{\mu A}^\mu - \partial_A \mathcal{H} = 0.$$

A key point is that these Hamilton equations coincide with the Euler–Lagrange equations of the first order Lagrangian

$$L_H = (p_A^\mu z_\mu^A - \mathcal{H}) \quad (41)$$

on the jet manifold $J^1 \Pi$ of $\Pi \rightarrow Q$. Indeed, its variational derivatives are

$$\mathcal{E}_\mu^A = z_\mu^A - \partial_\mu^A \mathcal{H}, \quad \mathcal{E}_A = -p_{\mu A}^\mu - \partial_A \mathcal{H}. \quad (42)$$

Any Hamiltonian form H (40) on Π yields the Hamiltonian map

$$\widehat{H} : \Pi \xrightarrow{Z_Q} J^1 Z_Q, \quad z_\mu^A \circ \widehat{H} = \partial_\mu^A \mathcal{H}. \quad (43)$$

A Hamiltonian \mathcal{H} on Π is said to be associated to a Lagrangian L on $J^1 Z_Q$ if it satisfies the relations

$$p_A^\mu = \partial_A^\mu \mathcal{L}(q^\nu, z^B, z_\lambda^B = \partial_\lambda^B \mathcal{H}), \quad (44)$$

$$p_A^\mu \partial_\mu^A \mathcal{H} - \mathcal{H} = \mathcal{L}(q^\nu, z^B, z_\lambda^B = \partial_\lambda^B \mathcal{H}). \quad (45)$$

If an associated Hamiltonian \mathcal{H} exists, the Lagrangian constraint space N_L is given by the coordinate equalities (44). The relation between Lagrangian and polysymplectic Hamiltonian formalisms is based on the following facts.

(i) Let a Lagrangian L be almost regular, and let us assume that it admits an associated Hamiltonian \mathcal{H} , which however need not be unique, unless L is hyperregular. In this case, the Poincaré–Cartan form H_L (26) is the pull-back $H_L = \hat{L}^*H$ of the Hamiltonian form H (40) for any associated Hamiltonian \mathcal{H} . Note that a local associated Hamiltonian always exists. The Poincaré–Cartan form H is a Lepagean equivalent both of the original Lagrangian L on J^1Z_Q and the Lagrangian

$$\overline{L} = (\mathcal{L} + (\hat{z}_\mu^A - z_\mu^A)\partial_A^\mu \mathcal{L})\omega$$

on the repeated jet manifold $J^1J^1Z_Q$. Its Euler–Lagrange equations are the Cartan equations for L . Any solution of the Euler–Lagrange equations (22) for L is also a solution of the Cartan equations. Furthermore, Euler–Lagrange equations and the Cartan equations are equivalent in the case of a regular Lagrangian.

(ii) If a Lagrangian L is almost regular, all associated Hamiltonian forms H coincide with each other on the Lagrangian constraint space N_L , and define the constrained Lagrangian $L_N = h_0(i_N^*H)$ on the jet manifold J^1N_L of the fiber bundle $N_L \rightarrow Q$. The Euler–Lagrange equations for this Lagrangian are called the constrained Hamilton equations. In fact, the Lagrangian L_H (41) is defined on the bundle product

$$\Pi \times_{Z_Q} J^1Z_Q, \quad (46)$$

and the constrained Lagrangian L_N is the restriction of L_H to $N_L \times J^1Z_Q$.

As a result, one can show that a section \overline{S} of the jet bundle $J^1Z_Q \rightarrow Q$ is a solution of the Cartan equations for L iff $\hat{L} \circ \overline{S}$ is a solution of the constrained Hamilton equations. In particular, any solution r of the constrained Hamilton equations provides the solution $\overline{S} = \hat{H} \circ r$ of the Cartan equations.

Turn now to symmetries of a Lagrangian L_H (41). Any vector field u on Z_Q gives rise to the vector field

$$u_\Pi = u^\mu \partial_\mu + u^A \partial_A + (-\partial_A u^B p_B^\mu - \partial_\lambda u^\lambda p_A^\mu + \partial_\lambda u^\mu p_A^\lambda) \partial_\mu^A$$

onto the Legendre bundle Π . Then we obtain its prolongation

$$u_\Pi = u^\mu \partial_\mu + u^A \partial_A + (d_\mu u^A - z_\nu^A d_\mu u^\nu) \partial_A^\mu + (-\partial_A u^B p_B^\mu - \partial_\lambda u^\lambda p_A^\mu + \partial_\lambda u^\mu p_A^\lambda) \partial_\mu^A$$

onto the product (46). It is a variational symmetry of the Lagrangian L_H if the Lie derivative $\mathbf{L}_{u_\Pi} L_H$ is d_H -exact.

For instance, let $u = u^\mu \partial_\mu$ be an arbitrary vector field on Q . Since $Z_Q \rightarrow Q$ is a trivial bundle, this vector field gives rise to a vector field $u = u^\mu \partial_\mu$ on Z_Q whose lift onto the Legendre bundle Π is

$$u_\Pi = u^\mu \partial_\mu + (-\partial_\lambda u^\lambda p_A^\mu + \partial_\lambda u^\mu p_A^\lambda) \partial_\mu^A.$$

Then we obtain its prolongation

$$u_\Pi = u^\mu \partial_\mu - z_\nu^A \partial_\mu u^\nu \partial_A^\mu + (-\partial_\lambda u^\lambda p_A^\mu + \partial_\lambda u^\mu p_A^\lambda) \partial_\mu^A$$

onto the product (46), and take its vertical part

$$u_V = -u^\nu z_\nu^A \partial_A - d_\mu(u^\nu z_\nu^A) \partial_A^\mu + (-\partial_\lambda u^\lambda p_A^\mu + \partial_\lambda u^\mu p_A^\lambda - u^\nu p_{\nu A}^\mu) \partial_\mu^A. \quad (47)$$

Let us regard it as a generalized vector field dependent on parameter functions $u^\mu(q)$. In accordance with Proposition 1, let us require that a Lagrangian L_H is independent on coordinates on Q and possesses the gauge symmetry u_V (47). Then its variational derivatives (42) of L_H obey the Noether identities

$$z_\nu^A \mathcal{E}_A + p_{\mu A}^\mu \mathcal{E}_\nu^A + p_A^\mu (d_\mu \mathcal{E}_\nu^A - d_\nu \mathcal{E}_\mu^A) = 0,$$

which reduce to rather restrictive conditions

$$\boxed{\delta_\nu^\mu \mathcal{H} = (n-1) p_A^\mu \partial_\nu^A \mathcal{H}} \quad (48)$$

which a Hamiltonian \mathcal{H} must satisfy. For instance, $\mathcal{H} = 0$ if $n = 1$. In this case, momenta are scalars relative to transformations of q and, therefore, no function of them is a density with respect to these transformations.

5 Example. $n = 1, 2$

Given an m -dimensional manifold Z coordinated by (z^A) , let us consider the jet manifold $J_1^1 Z$ of its one-dimensional submanifolds. Let us provide $Z = Z_1^0$ with coordinates $(x^0 = z^0, y^i = z^i)$ (3). Then the jet manifold $J_1^1 Z$ is endowed with coordinates (z^0, z^i, z_0^i) possessing transition functions (4), (8) which read

$$z'^0 = z'^0(z^0, z^k), \quad z'^0 = z'^0(z^0, z^k), \quad z_0'^i = \left(\frac{\partial z'^i}{\partial z^j} z_0^j + \frac{\partial z'^i}{\partial z^0} \right) \left(\frac{\partial z'^0}{\partial z^j} z_0^j + \frac{\partial z'^0}{\partial z^0} \right)^{-1}. \quad (49)$$

A glance at the transformation law (49) shows that $J_1^1 Z \rightarrow Z$ is a fiber bundle in projective spaces.

For instance, put $Z = \mathbb{R}^4$ whose Cartesian coordinates are subject to the Lorentz transformations

$$z'^0 = z^0 \text{ch}\alpha - z^1 \text{sh}\alpha, \quad z'^1 = -z^0 \text{sh}\alpha + z^1 \text{ch}\alpha, \quad z'^{2,3} = z^{2,3}. \quad (50)$$

Then z'^i (49) are exactly the Lorentz transformations

$$z_0'^1 = \frac{z_0^1 \text{ch}\alpha - \text{sh}\alpha}{-z_0^1 \text{sh}\alpha + \text{ch}\alpha} \quad z_0'^{2,3} = \frac{z_0^{2,3}}{-z_0^1 \text{sh}\alpha + \text{ch}\alpha}$$

of three-velocities in relativistic mechanics [14, 15].

Let us consider a one-dimensional manifold $Q = \mathbb{R}$ and the product $Z_Q = \mathbb{R} \times Z$. Let \mathbb{R} be provided with a Cartesian coordinate τ possessing transition function $\tau' = \tau + \text{const}$, unless otherwise stated. Then the jet manifold $J^1 Z_Q$ of the fiber bundle $\mathbb{R} \times Z \rightarrow Z$ is endowed with the coordinates $(\tau, z^0, z^i, z_\tau^0, z_\tau^i)$ with the transition functions

$$z_\tau'^0 = \frac{\partial z'^0}{\partial z^k} z_\tau^k + \frac{\partial z'^0}{\partial z^0} z_\tau^0, \quad z_\tau'^i = \frac{\partial z'^i}{\partial z^k} z_\tau^k + \frac{\partial z'^i}{\partial z^0} z_\tau^0. \quad (51)$$

A glance at this transformation law shows that, unless nonadditive transformations of τ are considered, there is an isomorphism

$$J_1^1 Z_Q = V Z_Q = \mathbb{R} \times T Z \quad (52)$$

of the jet manifold $J_1^1 Z_Q$ to the vertical tangent bundle $V Z_Q$ of $Z_Q \rightarrow \mathbb{R}$ which, in turn, is a product of \mathbb{R} and the tangent bundle $T Z$ of Z .

Returning to the example of $Z = \mathbb{R}^4$ and Lorentz transformations (50), one easily observed that transformations (51) are transformations of four-velocities in relativistic mechanics where τ is a proper time.

Let us consider coordinate charts $(U'; \tau, z^0, z^i, z_0^i)$ and $(U''; \tau, z^0, z^i, z_\tau^0, z_\tau^i)$ of the manifolds $\mathbb{R} \times J_1^1 Z$ and $J^1 Z_Q$ over the same coordinate chart $(U; \tau, z^0, z^i)$ of Z_Q . Then one can associate to each element (τ, z^0, z^i, z_0^i) of $U' \subset \mathbb{R} \times J_1^1 Z$ the elements of $U'' \subset J^1 Z_Q$ which obey the relations

$$z_0^i z_\tau^0 = z_\tau^i \quad (53)$$

and, in particular, the relations

$$z_0^i = \frac{z_\tau^i}{z_\tau^0}, \quad z_\tau^0 \neq 0. \quad (54)$$

Given a point $(\tau, z) \in \mathbb{R} \times Z$, the relations (53) – (54) are exactly the correspondence between elements of a one-dimensional vector subspace of the tangent space $T_z Z$ and the corresponding element of the projective space of these subspaces.

In the above mentioned example of relativistic mechanics, the relations (53) – (54) are familiar equalities between three- and four-velocities. It should be emphasized that, in relativistic mechanics, one avoids the ambiguity between three- and four-velocities by means of the nonholonomic constraint

$$(z_\tau^0)^2 - \sum_i (z_\tau^i)^2 = 1. \quad (55)$$

In a general setting, Lagrangian formalism on the jet manifold $J^1 Z_Q$ can be developed if a Lagrangian L is independent of τ , and it is variationally invariant under transformations reparametrizations $\tau'(\tau)$, i.e., its Euler–Lagrange operator obeys the Noether identity

$$z_\tau^A \mathcal{E}_A = 0. \quad (56)$$

For instance, let Z be a locally affine manifold, i.e., a toroidal cylinder $\mathbb{R}^{m-k} \times T^k$. Its tangent bundle can be provided with a constant nondegenerate fiber metric η_{AB} . Then

$$L = (\eta_{AB} z_\tau^A z_\tau^B)^{1/2} d\tau \quad (57)$$

is a Lagrangian on $J^1 Z_Q$. It is easily justified that this Lagrangian satisfies the Noether identity (56). Furthermore, given a one-form $\mathcal{A}_B dz^B$ on Z , one can consider the Lagrangian

$$L' = [(\eta_{AB} z_\tau^A z_\tau^B)^{1/2} - \mathcal{A}_B z_\tau^B] d\tau, \quad (58)$$

which also obeys the Noether identity (56). In relativistic mechanics, the Euler–Lagrange equations of the Lagrangians L (57) and L' (58) restricted to the constraint space (55) restart the familiar equations of motion of a free relativistic particle and a relativistic particle in the presence of an electromagnetic field \mathcal{A} .

As was mentioned above, no Hamiltonian obeys the Noether identities (48) if $n = 1$. However, Hamiltonian relativistic mechanics can be developed in the framework of Hamiltonian theory of mechanical systems with nonholonomic constraints [15, 23, 24]. A key is that the constraint condition (55) is not preserved under transformations of τ , and a Hamiltonian of a mechanical system with this constraint need not satisfy the Noether identities (48).

In comparison with the case of one-dimensional submanifolds, a description of the Lagrangian and Hamiltonian dynamics of two-dimensional submanifolds follows general theory of n -dimensional submanifolds. This is the case of classical string theory [25, 26, 27].

For instance, let Z be again an m -dimensional locally affine manifold, i.e., a toroidal cylinder $\mathbb{R}^{m-k} \times T^k$, and let Q be a two-dimensional manifold. As was mentioned above, the tangent bundle of Z can be provided with a constant nondegenerate fiber metric η_{AB} . Let us consider the 2×2 matrix with the entries

$$h_{\mu\nu} = \eta_{AB} z_\mu^A z_\nu^B.$$

Then its determinant provides a Lagrangian

$$L = (\det h)^{1/2} d^2 q = ([\eta_{AB} z_1^A z_1^B][\eta_{AB} z_2^A z_2^B] - [\eta_{AB} z_1^A z_2^B]^2)^{1/2} d^2 q \quad (59)$$

on the jet manifold $J^1 Z_Q$ (13). This is the well known Nambu–Goto Lagrangian of string theory. It satisfies the Noether identities (31). Let

$$F = \frac{1}{2} F_{AB} dz^A \wedge dz^B$$

be a two-form on a manifold Z . Then

$$F = \frac{1}{2} F_{AB} z_\mu^A z_\nu^B dq^\mu \wedge dq^\nu$$

is a horizontal density on $J^1 Z_Q$ which can be treated as an interaction term of submanifolds and an external classical field F in a Lagrangian.

Turn now to Hamiltonian theory of two-dimensional submanifolds on the momentum phase space Π (37). In this case, the Noether identities (48) take the form

$$\delta_\nu^\mu \mathcal{H} = p_A^\mu \partial_\nu^A \mathcal{H}. \quad (60)$$

For instance, let Z be the above mentioned toroidal cylinder whose cotangent bundle is provided with a constant nondegenerate fiber metric η_{AB} . Let us consider the 2×2 matrix with the entries

$$H^{\mu\nu} = \eta^{AB} p_A^\mu p_B^\nu.$$

Then its determinant provides a Hamiltonian

$$\mathcal{H} = (\det H)^{1/2} d^2 q = ([\eta^{AB} p_A^1 p_B^1][\eta^{AB} p_A^2 p_B^2] - [\eta^{AB} p_A^1 p_B^2]^2)^{1/2} d^2 q$$

on the momentum phase space Π which satisfies the Noether identities (60). This Hamiltonian is associated to the Lagrangian (59).

References

- [1] F.Takens, A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543.
- [2] I.Anderson, Introduction to the variational bicomplex, *Contemp. Math.*, **132** (1992) 51.
- [3] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Cohomology of the infinite-order jet space and the inverse problem, *J. Math. Phys.* **42** (2001) 4272.
- [4] G.Barnish, F.Brandt and M.Henneaux, Local BRST cohomology in gauge theories, *Phys. Rep.* **338** (2000) 439.
- [5] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology. *Commun. Math. Phys.* **259** (2005) 103; *E-print arXiv*: math.AG/0305303.
- [6] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether's second theorem for BRST symmetries. *J. Math. Phys.* **46** (2005) 053517; *E-print arXiv*: math-ph/0412034.
- [7] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, Singapore, 1997).
- [8] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Covariant Hamilton equations for field theory, *J. Phys. A* **32** (1999) 6629; *E-print arXiv*: hep-th/9904062.
- [9] O.Krupkova, Hamiltonian field theory, *J. Geom. Phys.* **43** (2002) 93.
- [10] A.Echeverría Enríquez, G.López, J.Marin-Solano, M.Muñoz Lecanda and N.Román Roy, Lagrangian-Hamiltonian unified formalism for field theories, *J. Math. Phys.* **45** (2004) 360.
- [11] M de Leon, D.Martín de Diego and A.Santamaría-Merini, Symmetries in classical field theory, *Int. J. Geom. Methods. Mod. Phys.* **1** (2004) 651.
- [12] I.Krasil'shchik, V.Lychagin and A.Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and breach, Glasgow, 1985)

- [13] M.Modugno and A.Vinogradov, Some variations on the notion of connections, *Ann. Matem. Pura ed Appl.* **CLXVII** (1994) 33.
- [14] G.Sardanashvily, Hamiltonian time-dependent mechanics, *J. Math. Phys.* **39** (1998) 2714.
- [15] L.Mangiarotti and G.Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).
- [16] N.Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, 1972).
- [17] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether's second theorem in a general setting. Reducible gauge theories, *J. Phys. A* **38** (2005) 5329; *E-print arXiv: math.DG/0411070*
- [18] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, The antifield Koszul–Tate complex of reducible Noether identities. *J. Math. Phys.* **46** (2005) 103513; *E-print arXiv: math-ph/0506034*.
- [19] G.Sardanashvily, Constraint field systems in multimomentum canonical variables, *J. Math. Phys.* **35** (1994) 6584.
- [20] G.Sardanashvily, *Generalized Hamiltonian Formalism for Field theory* (World Scientific, Singapore, 1995).
- [21] D.Saunders, *The Geometry of Jet Bundles* (Cambr. Univ. Press, Cambridge, 1989).
- [22] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).
- [23] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Nonholonomic constraints in time-dependent mechanics *J. Math. Phys.* **40** (1999) 1375; *E-print arXiv: math-ph/9807014*.
- [24] G.Sardanashvily, Geometric quantization of relativistic Hamiltonian mechanics, *Int. J. Theor. Phys.* **42** (2003) 697; *E-print arXiv: gr-qc/0208073*.
- [25] J.Scherk, An introduction to the theory of dual models and strings, *Rev. Mod. Phys.* **47** (1975) 123.

- [26] B.Hatfield, *Quantum Field Theory of Point Particles and Strings* (Addison–Wiley Publ., Redwood City, CA, 1992).
- [27] J.Polchinski, *String Theory* (Cambr. Univ. Press, Cambridge, 1998).